

A p -LAPLACIAN NEUMANN PROBLEM WITH A POSSIBLY SUPERCRITICAL NONLINEARITY

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ABSTRACT. We look for nonconstant, positive, radially nondecreasing solutions of the quasilinear equation $-\Delta_p u + u^{p-1} = f(u)$ with $p > 2$, in the unit ball B of \mathbb{R}^N , subject to homogeneous Neumann boundary conditions. The assumptions on the nonlinearity f are very mild and allow it to be possibly supercritical in the sense of Sobolev embeddings. The main tools used are the truncation method and a mountain pass-type argument. In the pure power case, i.e., $f(u) = u^{q-1}$, we detect the limit profile of the solutions of the problems as $q \rightarrow \infty$.

1. INTRODUCTION AND MAIN RESULTS

In [3], we study the existence of nonconstant, radially nondecreasing solutions of the following quasilinear problem

$$\begin{cases} -\Delta_p u + u^{p-1} = f(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \quad (1.1)$$

where B is the unit ball of \mathbb{R}^N , $N \geq 1$, ν is the outer unit normal of ∂B , and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator, with $p > 2$. We require very mild assumptions on the nonlinearity f on the right-hand side, namely $f \in C^1([0, \infty))$ and satisfies the following hypotheses

- (f₁) $\lim_{s \rightarrow 0^+} \frac{f(s)}{s^{p-1}} \in [0, 1)$;
- (f₂) $\liminf_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} > 1$;
- (f₃) \exists a constant $u_0 > 0$ such that $f(u_0) = u_0^{p-1}$ and $f'(u_0) > (p-1)u_0^{p-2}$.

Our main results in [3] read as follows.

Theorem 1.1. *If f satisfies (f₁)-(f₃), there exists a nonconstant, radially nondecreasing solution of (1.1). If furthermore there exist n different positive constants $u_0^{(1)} \neq \dots \neq u_0^{(n)}$ for which (f₃) holds, then (1.1) admits at least n distinct nonconstant, radially nondecreasing solutions.*

Theorem 1.2. *Let $f(u) = u^{q-1}$, with $q > p$. Denote by u_q the solution found in Theorem 1.1, corresponding to such f . Then, as $q \rightarrow \infty$,*

$$u_q \rightarrow G \text{ in } W^{1,p}(B) \cap C^{0,\mu}(\overline{B}) \quad \text{for any } \mu \in (0, 1),$$

where G is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_p G + G^{p-1} = 0 & \text{in } B, \\ G = 1 & \text{on } \partial B. \end{cases}$$

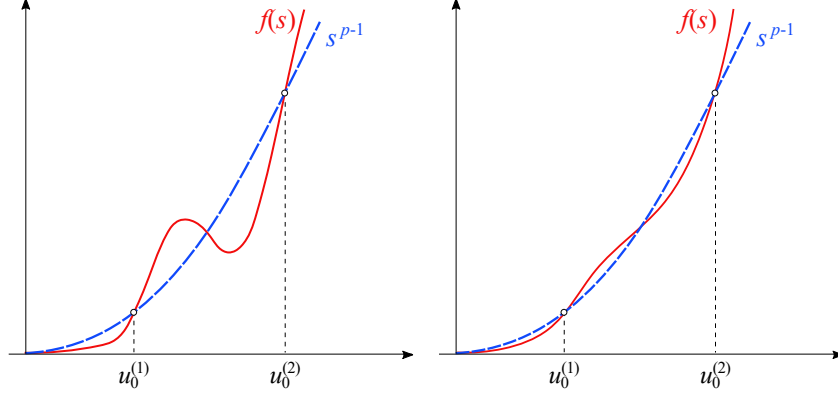


FIGURE 1. *Left:* Graph of a sample nonlinearity f satisfying (f_1) – (f_3) .
Right: Graph of a sample nonlinearity f satisfying (f_0) – (f_3) .

REMARKS.

- We observe that f is allowed to be supercritical in the sense of Sobolev embeddings, which will be the most interesting case.
- The model f is the pure power function $f(u) = u^{q-1}$, with $q > p$. In this case, problem (1.1) admits the constant solution $u \equiv 1$ for every $q > p$, including the supercritical case $q > p^*$, where $p^* := Np/(N-p)$ if $p < N$ and $p^* := +\infty$ otherwise. Therefore, the natural question that arises is whether (1.1) admits any *nonconstant* solutions. It is worth stressing a remarkable difference between problem (1.1) and the analogous problem under homogeneous Dirichlet boundary conditions. Indeed, it is well-known that, as a consequence of the Pohožaev identity (cf. [5, Section 2]), the Dirichlet problem does not admit any nonzero solutions when $q \geq p^*$.
- We remark that condition (f_3) is absolutely natural under (f_1) and (f_2) . Indeed, by the regularity of f and by (f_1) – (f_2) , there must exist an intersection point u_0 between f and the power s^{p-1} such that $f'(u_0) \geq (s^{p-1})'(u_0) = (p-1)u_0^{p-2}$. Hence, (f_3) is only meant to exclude the possibility of a degenerate situation in which f is tangent to s^{p-1} at u_0 .
- We can always think f to satisfy also
 $(f_0) \quad f \geq 0 \text{ and } f' \geq 0.$

Indeed, if this is not the case, we can replace f by $g(s) := f(s) + (m-1)s^{p-1}$ for a suitable $m > 1$ such that $g \geq 0$ and $g' \geq 0$, and study the equivalent problem

$$\begin{cases} -\Delta_p u + mu^{p-1} = g(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } B. \end{cases}$$

Therefore, without loss of generality, *from now on in the paper we assume f to satisfy (f_0) as well.*

Since f is possibly supercritical, the energy functional I associated to the problem is not well defined in the whole of $W^{1,p}(B)$, and so a priori we cannot use variational techniques to solve the problem. This issue is overcome for the first time in [6] for the semilinear case ($p = 2$) and then in [7] for any $1 < p < \infty$, by working in the

closed and convex cone

$$\mathcal{C} := \left\{ u \in W_{\text{rad}}^{1,p}(B) : u \geq 0 \text{ and } u(r) \leq u(s) \text{ for } r \leq s \right\},$$

where we have denoted by $W_{\text{rad}}^{1,p}(B)$ the space of $W^{1,p}(B)$ -functions which are radially symmetric and with abuse of notation we have written $u(x) = u(r)$ for $|x| = r$. Indeed, this cone has the property that all its functions are bounded, i.e.,

$$\|u\|_{L^\infty(B)} \leq C(N)\|u\|_{W^{1,p}(B)} \quad \text{for some } C(N) > 0 \text{ independent of } u \in \mathcal{C}, \quad (1.2)$$

see e.g. [3, Lemma 2.2]. Due to (1.2), it makes sense to define an energy functional I in \mathcal{C} , associated to the equation. On the other hand, the main disadvantage for working in this cone is the fact that it has empty interior in the $W^{1,p}$ -topology. As a consequence, in general, critical points of $I : \mathcal{C} \rightarrow \mathbb{R}$ are not solutions of (1.1). In [6, 7], the authors require additional assumptions on f to prove that the critical point of I , found via variational techniques, is indeed a weak solution of the problem. While in [2], in order to weaken the hypotheses on f , a different strategy based on the truncation method is proposed.

The techniques that we use in [3] to prove Theorem 1.1 are essentially in the spirit of [2]. The scheme of the proof can be split into five steps.

STEP 1. We first obtain, in [3, Lemma 2.5], the following a priori estimate

$$\|u\|_{L^\infty(B)} \leq K_\infty \quad \text{for all } u \in \mathcal{C} \text{ that solves (1.1),}$$

for some $K_\infty > 0$ independent of u . Clearly, $K_\infty \geq u_0$, being $u \equiv u_0$ a solution of (1.1) belonging to \mathcal{C} .

STEP 2. This allows us to truncate the nonlinearity f , in order to deal with a subcritical nonlinearity \tilde{f} , and so in [3, Lemma 3.1], we prove that

$$\begin{aligned} &\text{For all } \ell \in (p, p^*) \text{ there exists } \tilde{f} \in C^1([0, \infty)) \text{ satisfying } (f_0)-(f_3), \\ &\lim_{s \rightarrow \infty} \frac{\tilde{f}(s)}{s^{\ell-1}} = 1, \quad \text{and} \quad \tilde{f} = f \text{ in } [0, K_\infty]. \end{aligned}$$

We introduce the following auxiliary problem

$$\begin{cases} -\Delta_p u + u^{p-1} = \tilde{f}(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B. \end{cases} \quad (1.3)$$

As a consequence of the previous two steps, it is immediate to see that

In the cone \mathcal{C} , the two problems (1.1) and (1.3) are equivalent.

STEP 3. Thanks to the subcriticality of \tilde{f} , we can define the energy functional associated to (1.3) in the whole of $W^{1,p}(B)$ as follows

$$\tilde{I}(u) := \frac{1}{p} \int_B (|\nabla u|^p + |u|^p) dx - \int_B \tilde{F}(u) dx, \quad \text{where } \tilde{F}(u) := \int_0^u \tilde{f}(s) ds$$

for all $u \in W^{1,p}(B)$. All critical points of \tilde{I} are weak solutions of (1.3).

Remark 1.3. Since $p > 2$, \tilde{I} is of class C^2 , while if $1 < p < 2$, the functional \tilde{I} is only of class C^1 . This lack of regularity prevents either the use of second order Taylor expansions as done in [2, 3] (see also Section 3 below) or the use of a generalized Morse Lemma when looking for *nonconstant* solutions. Moreover, when $1 < p < 2$, Simon's inequalities relating \tilde{I}' and the pseudo-differential gradient are

weaker than the ones found for the case $p > 2$, this makes harder the construction of a descending flow and consequently the proof of a deformation-type lemma.

STEP 4. We find a critical point u of \tilde{I} belonging to \mathcal{C} via a mountain pass-type argument. We localize the solution in such a way that, if we have n different positive constants $u_0^{(i)}$ verifying (f_3) , we get “for free” also the multiplicity result stated in Theorem 1.1.

STEP 5. We prove that the solution found in Step 4. is nonconstant, by using a second order Taylor expansion of \tilde{I} .

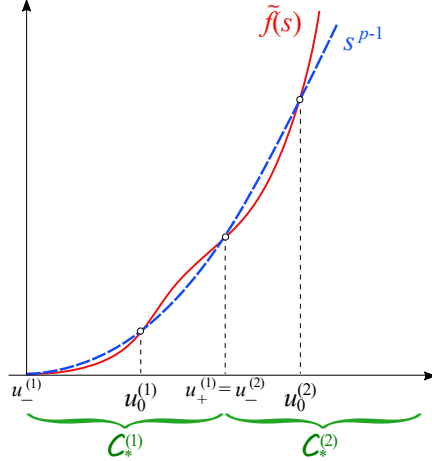
In the next two sections we give some details about Steps 4. and 5., respectively. While in the last section we sketch the proof of Theorem 1.2.

2. STEP 4: A NONCONSTANT SOLUTION OF (1.1) BELONGING TO \mathcal{C}

Due to the subcriticality of \tilde{f} , it is standard to prove the following compactness result (cf. [3, Lemma 3.4]):

The functional \tilde{I} satisfies the Palais-Smale condition.

The restricted cone \mathcal{C}_* .



Let $n \in \mathbb{N}$ be the number of positive constants $u_0^{(i)}$ satisfying (f_3) . For every $i = 1, \dots, n$, we set

$$u_-^{(i)} := \sup \left\{ s \in [0, u_0^{(i)}) : \tilde{f}(s) = s^{p-1} \right\},$$

$$u_+^{(i)} := \inf \left\{ s \in (u_0^{(i)}, \infty) : \tilde{f}(s) = s^{p-1} \right\}.$$

For every i , we further introduce the following subset of \mathcal{C}

$$\mathcal{C}_*^{(i)} := \left\{ u \in \mathcal{C} : u_-^{(i)} \leq u \leq u_+^{(i)} \right\}$$

which turns out to be itself a closed convex cone of $W^{1,p}(B)$.

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- Thanks to (f_3) , each $u_0^{(i)}$ is an isolated zero of $\tilde{f}(s) - s^{p-1}$, hence $u_-^{(i)} \neq u_0^{(i)} \neq u_+^{(i)}$ for every $i = 1, \dots, n$.
- We observe that $u_+^{(n)}$ can be possibly $+\infty$. For instance, for the pure power function $f(u) = u^{q-1}$ with $q > p$, it results $n = 1$, $u_- = 0$, $u_0 = 1$, $u_+ = +\infty$, and $\mathcal{C} = \mathcal{C}_*$.
- All and only the zeros of $\tilde{f}(s) - s^{p-1}$ are constant solutions of (1.3), and so of (1.1). Hence, the only constant solutions of (1.1) belonging to $\mathcal{C}_*^{(i)}$ are $u_-^{(i)}$, $u_0^{(i)}$, and $u_+^{(i)}$.
- If we prove the existence of a nonconstant solution u belonging to $\mathcal{C}_*^{(i)}$, we know at once that $u_-^{(i)} \leq u \leq u_+^{(i)}$ and that $u_-^{(i)} \neq u \neq u_+^{(i)}$. This implies that nonconstant solutions of (1.1) belonging to different $\mathcal{C}_*^{(i)}$'s are different.

As a consequence of the last two remarks, we can see that the advantage of working in $\mathcal{C}_*^{(i)}$ instead of \mathcal{C} is twofold. Firstly, it helps avoiding constant solutions:

it is enough to prove that the solution found is none of the three constant solutions in $\mathcal{C}_*^{(i)}$. Secondly, the restricted cone $\mathcal{C}_*^{(i)}$ allows us to localize our solution, so that the multiplicity part of Theorem 1.1 follows immediately by the existence part.

Hereafter, we assume for simplicity $n = 1$ and we omit all the superscripts (i) . Clearly, if $n > 1$, it is possible to repeat the same arguments in each cone $\mathcal{C}_*^{(i)}$.

A deformation lemma. This is the most technical part of the proof. Since the space $W^{1,p}(B)$ in which the energy functional \tilde{I} is defined is bigger than the set \mathcal{C}_* in which we want to find a minimax solution, we need a slightly different version of the deformation lemma.

Lemma 2.1 (Lemma 3.9 of [3]). *Let $c \in \mathbb{R}$ be such that $\tilde{I}'(u) \neq 0$ for all $u \in \mathcal{C}_*$, with $\tilde{I}(u) = c$. Then, there exist a positive constant $\bar{\varepsilon}$ and a function $\eta : \mathcal{C}_* \rightarrow \mathcal{C}_*$ satisfying the following properties:*

- (i) η is continuous with respect to the topology of $W^{1,p}(B)$;
- (ii) $I(\eta(u)) \leq I(u)$ for all $u \in \mathcal{C}_*$;
- (iii) $I(\eta(u)) \leq c - \bar{\varepsilon}$ for all $u \in \mathcal{C}_*$ such that $|I(u) - c| < \bar{\varepsilon}$;
- (iv) $\eta(u) = u$ for all $u \in \mathcal{C}_*$ such that $|I(u) - c| > 2\bar{\varepsilon}$.

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- We stress here that we build a deformation η not only for regular values c of \tilde{I} (i.e., such that $\tilde{I}'(u) \neq 0$ for all $u \in W^{1,p}(B)$ with $\tilde{I}(u) = c$), but also for all $c \in \mathbb{R}$ for which $\tilde{I}'(u) \neq 0$ for all $u \in \mathcal{C}_*$ with $\tilde{I}(u) = c$.
- In this version of the deformation lemma, we need to prove that the η preserves the cone \mathcal{C}_* . This is the most delicate point of the proof. It requires the existence of a pseudo-gradient vector field K of \tilde{I} which is not only locally Lipschitz continuous, but which satisfies also the following property

$$K\left(\mathcal{C}_* \setminus \{\text{critical points of } \tilde{I}\}\right) \subset \mathcal{C}_*. \quad (2.1)$$

Indeed, for every $u \in \mathcal{C}_*$, the deformation $\eta(u)$ is built as the unique solution $\mu(t, u)$ of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\mu(t, u(x)) = -\Phi(\mu(t, u(x))) & (t, x) \in (0, \infty) \times B, \\ \partial_\nu \mu(t, u(x)) = 0 & (t, x) \in (0, \infty) \times \partial B, \\ \mu(0, u(x)) = u(x) & x \in B, \end{cases}$$

where $\Phi(u) := \begin{cases} \chi_1(I(u))\chi_2(u) \frac{u-K(u)}{\|u-K(u)\|} & \text{if } |I(u) - c| \leq 2\bar{\varepsilon}, \\ 0 & \text{otherwise,} \end{cases} \quad \chi_1, \chi_2 \text{ cutoff}$ (2.2)

for t (fixed) sufficiently large (i.e., $\eta(u) := \mu(\bar{t}, u)$). The existence of such operator K and of its properties are proved in [3, Proposition 3.2 and Lemmas 3.5-3.8] (see also [1] for the case of an open cone) and passes through the study of an auxiliary operator \tilde{T} related to the inverse of $-\Delta_p(\cdot) + |\cdot|^{p-2}(\cdot)$. In particular, property (2.1) is a consequence of the fact that $\tilde{T}(\mathcal{C}_*) \subseteq \mathcal{C}_*$, that is proved –by hands– in [3, Lemma 3.5]. Finally, thanks to (2.1), the convexity, and the closedness of \mathcal{C}_* , we are able to prove that $\eta(\mathcal{C}_*) \subseteq \mathcal{C}_*$.

- Condition (iv) is an immediate consequence of the fact that μ solves the Cauchy problem (2.2). While, (ii) and (iii) rely essentially on Simon-type inequalities, that is to say relations between \tilde{I}' and K , see [3, Proposition 3.2 and Lemmas 3.6-3.8].

A mountain pass-type geometry.

Lemma 2.2 (Lemma 3.10 and formula (3.32) of [3]). *Let τ be a constant such that $0 < \tau < \min\{u_0 - u_-, u_+ - u_0\}$. Then there exists $\alpha > 0$ such that*

- (i) $\tilde{I}(u) \geq \tilde{I}(u_-) + \alpha$ for every $u \in \mathcal{C}_*$ with $\|u - u_-\|_{L^\infty(B)} = \tau$;
- (ii) if $u_+ < \infty$, then $\tilde{I}(u) \geq \tilde{I}(u_+) + \alpha$ for every $u \in \mathcal{C}_*$ with $\|u - u_+\|_{L^\infty(B)} = \tau$.

Furthermore,

- (iii) $\tilde{I}(t \cdot 1) \rightarrow -\infty$ as $t \rightarrow +\infty$.

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- If $u_+ = +\infty$, then (i) and (iii) are pretty much the classical conditions required for the mountain pass geometry centered at u_- .
- If $u_+ < +\infty$, then the roles played by u_- and u_+ are interchangeable, hence we prove that the points on the sphere $\partial B_\tau(u_-) := \{u \in \mathcal{C}_* : \|u - u_-\|_{L^\infty(B)} = \tau\}$ and those on $\partial B_\tau(u_+) := \{u \in \mathcal{C}_* : \|u - u_+\|_{L^\infty(B)} = \tau\}$ satisfy the same condition with respect to u_- and to u_+ , respectively. In this case, since $u_0 - u_- > \tau$ and $u_+ - u_0 > \tau$, then the two closed balls $\overline{B}_\tau(u_-)$ and $\overline{B}_\tau(u_+)$ are disjoint. Therefore, suppose –to fix ideas– that $\tilde{I}(u_-) \leq \tilde{I}(u_+)$. By (ii), for all $u \in \partial B_\tau(u_+)$ it results $\tilde{I}(u) \geq \tilde{I}(u_+) + \alpha$ and there exists u_- , for which

$$\|u_- - u_+\|_{L^\infty(B)} > \tau \quad \text{and} \quad \tilde{I}(u_-) \leq \tilde{I}(u_+).$$

- We remark that in (i) and (ii) it is possible to use the L^∞ -norm instead of the $W^{1,p}$ -norm, because \mathcal{C}_* -functions are bounded by (1.2). In particular, the use of the L^∞ -norm allows us to simplify the constants.

Existence of a solution of (1.1) in \mathcal{C}_* . Let τ and α be the constants introduced in the previous subsection,

$$U_- := \left\{ u \in \mathcal{C}_* : \tilde{I}(u) < \tilde{I}(u_-) + \frac{\alpha}{2}, \|u - u_-\|_{L^\infty(B)} < \tau \right\},$$

$$U_+ := \begin{cases} \left\{ u \in \mathcal{C}_* : \tilde{I}(u) < \tilde{I}(u_+) + \frac{\alpha}{2}, \|u - u_+\|_{L^\infty(B)} < \tau \right\}, & \text{if } u_+ < \infty, \\ \left\{ u \in \mathcal{C}_* : \tilde{I}(u) < \tilde{I}(u_-), \|u - u_-\|_{L^\infty(B)} > \tau \right\}, & \text{if } u_+ = \infty \end{cases}$$

the sets from/to which the admissible paths used to define the minimax level start/arrive,

$$\Gamma := \{\gamma \in C([0, 1]; \mathcal{C}_*) : \gamma(0) \in U_-, \gamma(1) \in U_+\}$$

the set of admissible paths, and

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \tilde{I}(\gamma(t)) \tag{2.3}$$

the minimax level.

By combining together the compactness condition, the mountain pass-type geometry of \tilde{I} , and the deformation lemma presented above, we are able to prove the following result.

Proposition 2.3 (Proposition 3.11 of [3]). *The value c defined in (2.3) is finite and there exists a critical point $u \in \mathcal{C}_* \setminus \{u_-, u_+\}$ of \tilde{I} such that $\tilde{I}(u) = c$ and $u > 0$. In particular, u is a weak solution of (1.1).*

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- We observe that, since every admissible path $\gamma \in \Gamma$ starts from $B_\tau(u_-)$ and arrives in $B_\tau(u_+)$, due to its continuity, it must cross the sphere $\partial B_\tau(u_-)$ (and also $\partial B_\tau(u_+)$ if $u_+ < +\infty$). Then, by Lemma 2.2-(i) (and also by (ii) if $u_+ < +\infty$),

$$\tilde{I}(u_-) < c < +\infty \quad (\text{and also } \tilde{I}(u_+) < c \text{ if } u_+ < +\infty).$$

This immediately excludes the possibility that the solution u is the constant u_- (or the constant u_+ when this latter is finite).

- By the maximum principle [8, Theorem 5], u is positive.

3. STEP 5: THE SOLUTION FOUND IS NONCONSTANT.

In this section we conclude the proof of Theorem 1.1. As already observed in Section 2, the multiplicity part of the theorem follows easily when one works in the restricted cone \mathcal{C}_* . Concerning the nonconstancy of the solution, we already know by Proposition 2.3 that the solution $u \in \mathcal{C}_*$, at level c , is neither the constant u_- nor the constant u_+ . It remains to show that $u \neq u_0$. In particular, we prove that $c = \tilde{I}(u) < \tilde{I}(u_0)$.

By the very definition of c , it is enough to find an admissible path $\bar{\gamma}$ such that

$$\max_{t \in [0,1]} \tilde{I}(\bar{\gamma}(t)) < \tilde{I}(u_0). \quad (3.1)$$

We sketch below the construction of such curve $\bar{\gamma} \in \Gamma$, see [3, Lemma 4.3] for more details.

- It is easy to see that there exist two positive numbers t_- and t_+ ($t_- < 1 < t_+$), such that $t_-u_0 \in U_-$ and $t_+u_0 \in U_+$.
- By (f_3) , the function $t \in [t_-, t_+] \mapsto \tilde{I}(tu_0)$ has a unique strict maximum point at $t = 1$. Hence,

$$\tilde{I}(tu_0) < \tilde{I}(u_0) \quad \text{for all } t \in [t_-, t_+] \setminus \{1\}.$$

- Let $v \in W_{\text{rad}}^{1,p}(B) \setminus \{0\}$ be nondecreasing and such that $\int_B v dx = 0$. For every $t \in [t_-, t_+]$, the function $s \in \mathbb{R} \mapsto \tilde{I}(t(u_0 + sv))$ is continuous. Therefore, by the previous step, we get for s in a neighborhood of 0

$$\tilde{I}(t(u_0 + sv)) < \tilde{I}(u_0) \quad \text{for all } t \in [t_-, t_+] \setminus [1 - \varepsilon, 1 + \varepsilon],$$

where $\varepsilon > 0$ is a sufficiently small constant.

- In order to have the same inequality also for t close to 1, we use condition (f_3) , the C^2 -regularity of \tilde{I} and the Implicit Function Theorem, see [3, Lemma 4.1]. This allows us to prove that u_0 is not a local minimum of the Nehari-type set

$$\mathcal{N}_* := \{u \in \mathcal{C}_* : \tilde{I}'(u)[u] = 0\}.$$

In particular, we prove that for all $s \in \mathbb{R}$ there exists a unique $\bar{t}_s > 0$ such that $\bar{t}_s(u_0 + sv) \in \mathcal{N}_*$ and \bar{t}_s is the unique maximum point of the map $t \in [1 - \varepsilon, 1 + \varepsilon] \mapsto \tilde{I}(t(u_0 + sv))$. Furthermore, by using a second order Taylor expansion of the energy functional and (f_3) , we obtain that for s in a neighborhood of 0

$$\tilde{I}(\bar{t}_s(u_0 + sv)) - \tilde{I}(u_0) = \frac{s^2}{2} \int_B [(p-1)u_0^{p-2} - \tilde{f}'(u_0)]v^2 dx + o(s^2) < 0.$$

Therefore, we get for s close to 0

$$\tilde{I}(t(u_0 + sv)) \leq \tilde{I}(\bar{t}_s(u_0 + sv)) < \tilde{I}(u_0) \quad \text{for all } t \in [1 - \varepsilon, 1 + \varepsilon].$$

- Clearly, for $\bar{s} > 0$ small enough, $t_-(u_0 + \bar{s}v) \in U_-$ and $t_+(u_0 + \bar{s}v) \in U_+$.

- By the convexity of \mathcal{C}_* , keeping in mind that $U_-, U_+ \subset \mathcal{C}_*$,

$$t(u_0 + \bar{s}v) \in \mathcal{C}_* \quad \text{for every } t \in [t_-, t_+].$$

- Hence, the curve $\bar{\gamma} : t \in [0, 1] \mapsto ((1-t)t_- + tt_+)(u_0 + \bar{s}v) \in \mathcal{C}_*$ belongs to Γ and satisfies (3.1).

4. SKETCH OF THE PROOF OF THEOREM 1.2

We denote by $u_q \in \mathcal{C}$ the nonconstant solution of

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{q-1} & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B \end{cases} \quad (4.1)$$

at minimax level c_q and by \tilde{I}_q the energy functional associated to the corresponding truncated problem. We describe below the main steps to prove Theorem 1.2, see for reference [3, Theorem 1.3] and also [4].

- In [3, Lemma 5.5], we find an a priori bound on u_q , uniform in q . Namely,

$$\|u_q\|_{C^1(\bar{B})} \leq C, \quad \text{with } C > 0 \text{ independent of } q \geq p+1.$$

Here we use the special form of f .

- This ensures the existence of a limit profile u_∞ for which

$$u_q \rightharpoonup u_\infty \text{ in } W^{1,p}(B) \quad \text{and} \quad u_q \rightarrow u_\infty \text{ in } C^{0,\mu}(\bar{B}) \quad \forall \mu \in (0, 1) \quad \text{as } q \rightarrow \infty.$$

Furthermore, $u_\infty(1) = 1$, see [3, Lemma 5.6].

Remark 4.1. By integrating over B the first equation of problem (4.1), we get $\int_B u_q^{p-1}(1 - u_q^{q-p})dx = 0$. Since $u_q > 0$, $u_q \not\equiv 1$, and $u_q' \geq 0$, it results

$$u_q(0) < 1 \quad \text{and} \quad u_q(1) > 1 \quad \text{for all } q \geq p+1.$$

Heuristically, where $u_q \leq \text{Const.} < 1$ (i.e., near the center of the ball B), $\lim_{q \rightarrow \infty} u_q^{q-1} = 0$. So, it is natural to expect that u_∞ solves $-\Delta_p u + u^{p-1} = 0$ at least in a neighborhood of the origin. On the other hand, in the region where $u_q \geq 1$ (i.e., in a neighborhood of ∂B), the same limit is an indeterminate form. This is somehow responsible of the fact that the boundary condition is not preserved in the limit. We further remark that, by Hopf's lemma, $\partial_\nu G > 0$ on ∂B , hence the $C^{0,\mu}(\bar{B})$ -convergence is optimal.

- We introduce the quantity $c_\infty := \inf \left\{ \frac{1}{p} \|u\|_{W^{1,p}(B)}^p : u \in \mathcal{C}, u|_{\partial B} = 1 \right\}$ and we show that $c_\infty = \inf \left\{ \frac{1}{p} \|u\|_{W^{1,p}(B)}^p : u \in W^{1,p}(B), u|_{\partial B} = 1 \right\}$. Furthermore, this infimum is *uniquely* achieved at G (via the Direct Method of the Calculus of Variations), see [3, Lemma 5.7].

- We show in [3, Lemma 5.8] that $c_\infty = \lim_{q \rightarrow \infty} c_q$. The proof relies mainly on the fact that the minimax level c_q in the cone coincides with a Nehari-type level in the cone (also here we use the fact that f is a pure power function), cf. [3, Lemma 5.4]. As a consequence, we get that c_∞ is attained at u_∞ and $\|u_q\|_{W^{1,p}(B)} \rightarrow \|u_\infty\|_{W^{1,p}(B)}$.

- By uniqueness, $u_\infty = G$ a.e. in B . Finally, the weak convergence ($u_q \rightharpoonup G$ in $W^{1,p}(B)$) together with the convergence of the norms ($\|u_q\|_{W^{1,p}(B)} \rightarrow \|G\|_{W^{1,p}(B)}$) guarantee that $u_q \rightarrow G$ in $W^{1,p}(B)$, by the uniform convexity of the space.

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